

Small deviations of sums of correlated stationary Gaussian sequences

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Abstract

We consider the small deviation probabilities (SDP) for sums of stationary Gaussian sequences. For the cases of constant boundaries and boundaries tending to zero, we obtain quite general results. For the case of the boundaries tending to infinity, we focus our attention on the discrete analogs of the fractional Brownian motion (FBM). It turns out that the lower bounds for the SDP can be transferred from the well studied FBM case to the discrete time setting under the usual assumptions that imply weak convergence while the transfer of the corresponding upper bounds necessarily requires a deeper knowledge of the spectral structure of the underlying stationary sequence.

Keywords: Fractional Brownian motion, fractional Gaussian noise, Gaussian process, small deviation probability, stationary Gaussian sequence, time series.

1 Introduction and main results

1.1 Introduction

The small deviation problem for a stochastic process consists in studying the probability that the process only has fluctuations below its natural scale. Small deviation probabilities play a fundamental role in many problems in probability and analysis, which is why there has been a lot of interest in small deviation problems in recent years, cf. the survey [12] and the literature compilation [14]. There are many connections to other questions such as the law of the iterated logarithm of Chung type, strong limit laws in statistics, metric entropy properties of linear operators, quantization, and several other approximation quantities for stochastic processes.

Our work heavily relies on the recently proved Gaussian correlation inequality [17, 9]; and we believe that this new tool can lead to the solution of other, formerly inaccessible problems in the area of small deviation probabilities.

In this paper, we study small deviations of sums of correlated stationary centered Gaussian sequences that are related to Fractional Brownian motion (FBM). Let us first recall FBM and its small deviation asymptotics.

FBM $(W_t^H)_{t \in \mathbb{R}}$ is a centered Gaussian process with covariance

$$\mathbb{E} W_t^H W_s^H = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}),$$

where $0 < H < 1$ is a constant parameter, called Hurst parameter. For $H = 1/2$ this is a usual Brownian motion. For any $0 < H < 1$, the process has stationary increments, but no independent increments (unless $H = 1/2$). Furthermore, it is an H -self-similar process. Finally, we recall the small deviation asymptotics for fractional Brownian motion W^H [11]

$$\ln \mathbb{P}\left\{ \sup_{0 \leq t \leq 1} |W^H(t)| \leq \varepsilon \right\} \sim -\kappa_H \varepsilon^{-1/H}, \quad \text{as } \varepsilon \rightarrow 0,$$

where the constant $\kappa_H \in (0, \infty)$ is not known explicitly unless $H = 1/2$ (and $\kappa_{1/2} = \pi^2/8$). Using the scaling property of FBM, this can be re-written as

$$\ln \mathbb{P}\left\{ \sup_{0 \leq t \leq N} |W^H(t)| \leq f_N \right\} \sim -\kappa_H N f_N^{-1/H}, \quad \text{as } N \rightarrow \infty, N^{-H} f_N \rightarrow 0. \quad (1)$$

In this paper, we consider the discrete-time analog of fractional Brownian motion. Let $(\xi_j)_{j \in \mathbb{N}}$ be a real valued stationary centered Gaussian sequence such that

$$\sum_{j=1}^n \sum_{k=1}^n \mathbb{E} \xi_j \xi_k \sim n^{2H} \ell(1/n), \quad (2)$$

with $0 < H < 1$ and ℓ slowly varying at zero. It is well-known ([21]) that (2) implies

$$\left(\frac{1}{n^H \ell(1/n)^{1/2}} \sum_{j=1}^{[nt]} \xi_j \right)_{t \geq 0} \Rightarrow (W_t^H)_{t \geq 0} \quad (3)$$

with fractional Brownian motion (W_t^H) . We remark that the same holds if the (ξ_j) are not necessarily Gaussian, but certain moment restrictions hold, [22].

The question to be studied is the “small deviation” rate of $S_n := \sum_{j=1}^n \xi_j$, i.e.

$$\mathbb{P}\left\{ \max_{n=1, \dots, N} |S_n| \leq f_N \right\}, \quad \text{as } N \rightarrow \infty, \quad (4)$$

where $f_N \ll N^H \ell(1/N)^{1/2}$. As can be seen from the convergence result, $(S_n)_{1 \leq n \leq N}$ has fluctuations of the scale $N^H \ell(1/N)^{1/2}$, so that indeed we deal with a small deviation question.

There are three regimes: if $f_N \rightarrow \infty$ the small deviation properties of (S_n) are indeed governed by the same quantities as for FBM (at least under

some regularity assumptions, which are shown to be necessary). On the other hand, for $f_N \rightarrow 0$, we deal with “very small” deviations, and the rate is completely independent of any relation to FBM; we shall prove rather general results here – in particular, unrelated to (2). In the intermediate case when f_N is constant (or bounded away from zero and infinity), the behavior is similar to the “very small” deviation regime, and the rate of decay of the small deviation probability is precisely exponential.

Let us mention some related work. The classical case of *independent* (ξ_j) was studied by Chung [6], Mogul’skii [15], and Pakshirajan [16]. In particular, Mogul’skii showed that if the (ξ_j) are i.i.d. centered variables with unit variance, $f_N \rightarrow \infty$ but $N^{-1/2} f_N \rightarrow 0$, then, in agreement with (1) for $H = 1/2$,

$$\ln \mathbb{P}\{\max_{1 \leq n \leq N} |S_n| \leq f_N\} \sim -\frac{\pi^2}{8} N f_N^{-2}.$$

This paper is structured as follows. Sections 1.2, 1.3, and 1.4 contain the main results for the three mentioned regimes, respectively. The proofs are given in the subsequent sections.

1.2 Small deviations related to FBM

We first deal with the regime

$$f_N \rightarrow \infty \quad \text{and} \quad f_N \ll N^H \ell(1/N)^{1/2}$$

in (4). Our first main result (Theorem 1) states that a lower bound holds as one would expect from (1). In order to formulate it, let us recall the definition of the adjoint of a slowly varying function (see [19, Section 1.6]): for a function $\tilde{\ell}$ that is slowly varying at infinity, an adjoint function (unique up to asymptotic equivalence) is a slowly varying function $L(\cdot)$ satisfying the relation

$$L(r) \tilde{\ell}(rL(r)) \rightarrow 1, \quad \text{as } r \rightarrow \infty. \quad (5)$$

Now we are ready to state our first main result.

Theorem 1 *Let $(\xi_j)_{j \in \mathbb{N}}$ be a real valued stationary centered Gaussian sequence such that (3) holds. If $f_N \rightarrow \infty$ and $f_N \ll N^H \ell(1/N)^{1/2}$ then*

$$\liminf_{N \rightarrow \infty} \frac{\ln \mathbb{P}\{\max_{1 \leq n \leq N} |S_n| \leq f_N\}}{N [f_N L(f_N)]^{-1/H}} \geq -\kappa_H, \quad (6)$$

where $L(\cdot)$ is a slowly varying function adjoint to the function $\tilde{\ell}(r) := \sqrt{\ell(r^{-1/H})}$ and κ_H is the constant from (1).

The proof of this theorem is given in Section 2.

We shall prove that the corresponding upper bound surprisingly does *not* hold in this generality. In order to obtain the upper bound, one has to assume more than only the weak convergence to FBM (see Theorem 3 below), as the following negative result shows:

Theorem 2 *For any $H \in (0, 1)$ and any sequence f_N such that $f_N \rightarrow \infty$ and $N f_N^{-1/H} \rightarrow \infty$, there exists a real valued stationary centered Gaussian sequence $(\xi_j)_{j \in \mathbb{N}}$ such that (2) holds with $\ell \equiv 1$ but we have*

$$\limsup_{N \rightarrow \infty} \frac{\ln \mathbb{P}\{\max_{1 \leq n \leq N} |S_n| \leq f_N\}}{f_N^{-1/H} N} = 0. \quad (7)$$

The proof of this theorem is given in Section 4.

In order to obtain the upper bound corresponding to (6), – instead of only assuming weak convergence to FBM – we make an assumption about the spectral measure of the sequence (ξ_j) .

As above, let $(\xi_j)_{j \in \mathbb{N}}$ be a real valued stationary centered Gaussian sequence, and denote by μ the spectral measure:

$$\mathbb{E} \xi_j \xi_k = \mathbb{E} \xi_{|j-k|} \xi_0 = \int_{[-\pi, \pi)} e^{i|j-k|u} \mu(du), \quad j, k \in \mathbb{N}.$$

The spectral measure μ has a (possibly vanishing) component that is absolutely continuous w.r.t. the Lebesgue measure. Let us denote by p its density, i.e. $\mu(du) =: p(u)du + \mu_s(du)$.

Recall that fractional Gaussian noise, defined by $\xi_j^{\text{FGN}} := W^H(j) - W^H(j-1)$, is a stationary centered Gaussian sequence and it has an absolutely continuous spectral measure with a density p_{FGN} . The latter has a singularity at zero (see e.g. [18]):

$$p_{\text{FGN}}(u) \sim m_H |u|^{1-2H}, \quad u \rightarrow 0,$$

where $m_H = \Gamma(2H+1) \sin(\pi H)/2\pi$.

We assume that the density p of the absolutely continuous component of μ satisfies

$$p(u) \sim m_H \ell(u) |u|^{1-2H}, \quad u \rightarrow 0, \quad (8)$$

where $\ell(\cdot)$ is a function slowly varying at zero. This means that the behavior of the density of the absolutely continuous part of the spectral measure of the sequence (ξ_j) is comparable to the spectral density of fractional Gaussian noise, up to the slowly varying function ℓ . It is well-known (also see (17) below) that (8) implies (2) and thus (3).

Our second main result can now be formulated as follows.

Theorem 3 *Let $(\xi_j)_{j \in \mathbb{N}}$ be a real valued stationary centered Gaussian sequence. Assume that the density of the absolutely continuous component of the spectral measure satisfies (8). If $N \rightarrow \infty$, $[L(f_N)f_N]^{-1/H}N \rightarrow \infty$, and $f_N \rightarrow \infty$, then*

$$\ln \mathbb{P}\{\max_{1 \leq n \leq N} |S_n| \leq f_N\} \sim -\kappa_H [L(f_N)f_N]^{-1/H}N,$$

where again $L(\cdot)$ is a slowly varying function adjoint to the function $\tilde{\ell}(r) = \sqrt{\ell(r^{-1/H})}$ and κ_H is the constant from (1).

The proof of this theorem is given in Section 3.

1.3 Very small deviations

As the next step, we look at the opposite regime where

$$f_N \rightarrow 0.$$

Let us fix the setup here as follows. As above, we consider a real valued stationary centered Gaussian sequence $(\xi_j)_{j \in \mathbb{N}}$ with spectral measure μ :

$$\mathbb{E} \xi_j \xi_k = \mathbb{E} \xi_{|j-k|} \xi_0 = \int_{[-\pi, \pi)} e^{i|j-k|u} \mu(du).$$

Further, we denote by p the density of the absolutely continuous component of μ . As before, we will study the sums $S_n := \sum_{j=1}^n \xi_j$.

It is well-known (see e.g. [5]) that the sequence (ξ_j) is linearly regular if and only if its spectral measure is absolutely continuous and its density p satisfies the Kolmogorov condition

$$\int_{-\pi}^{\pi} \ln p(u) du > -\infty. \quad (9)$$

In the following we do not need the notion of regularity directly but condition (9) emerges below.

Our main theorem gives the first two terms of the small deviation rate under assumption (9), i.e. in the presence of the regular component. This includes in particular fractional Gaussian noise and related sequences but does not depend on any precise relation such as (8).

Theorem 4 *Let $(\xi_j)_{j \in \mathbb{N}}$ be a real valued stationary centered Gaussian sequence with spectral measure μ and denote by p the (possibly vanishing) density of the absolutely continuous component of μ . For $f_N \rightarrow 0$ we have:*

$$\liminf_{N \rightarrow \infty} \frac{\ln \mathbb{P}\{\max_{1 \leq n \leq N} |S_n| \leq f_N\}}{N \ln f_N^{-1}} \geq -1.$$

If additionally condition (9) holds, then

$$\ln \mathbb{P}\left\{\max_{1 \leq n \leq N} |S_n| \leq f_N\right\} = N \ln f_N - N \left[\ln \pi + \frac{1}{4\pi} \int_{-\pi}^{\pi} \ln p(u) du \right] + o(N),$$

The proof of this theorem is given in Section 5. We remark that if condition (9) does not hold, various different asymptotics may arise. As an illustration, we mention a few examples with $p = 0$. Here, \approx means that the ratio of both quantities is bounded away from zero and infinity.

Example 5 If $\mu = \delta_0$, then

$$\mathbb{P}\left\{\max_{1 \leq n \leq N} |S_n| \leq f_N\right\} \approx \frac{f_N}{N}.$$

If $\mu = \delta_{-\pi}$, then

$$\mathbb{P}\left\{\max_{1 \leq n \leq N} |S_n| \leq f_N\right\} \approx f_N.$$

If $\mu = \delta_{-\pi/2} + \delta_{\pi/2}$, then

$$\mathbb{P}\left\{\max_{1 \leq n \leq N} |S_n| \leq f_N\right\} \approx f_N^2.$$

If $\mu = \delta_0 + \delta_{-\pi} + \delta_{\pi/2} + \delta_{-\pi/2}$, then

$$\mathbb{P}\left\{\max_{1 \leq n \leq N} |S_n| \leq f_N\right\} \approx \frac{f_N^4}{N}.$$

1.4 Constant boundary

Finally, we look at an intermediate regime including the case where $f_N = f$ is constant. The setup is the same as in Section 1.3: Consider a real valued stationary centered Gaussian sequence $(\xi_j)_{j \in \mathbb{N}}$ with spectral measure μ , set $S_n := \sum_{j=1}^n \xi_j$, and denote by p the density of the absolutely continuous component of μ .

Theorem 6 Let (f_N) be a positive sequence having a finite positive limit. Then the following limit exists:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln \mathbb{P}\left\{\max_{1 \leq n \leq N} |S_n| \leq f_N\right\} \in (-\infty, 0].$$

In particular, for every constant $f > 0$ the following limit exists:

$$\mathfrak{C}(f) := \lim_{N \rightarrow \infty} \frac{1}{N} \ln \mathbb{P}\left\{\max_{1 \leq n \leq N} |S_n| \leq f\right\} \in (-\infty, 0].$$

If additionally the Kolmogorov criterion (9) is satisfied, then $\mathfrak{C}(f) < 0$.

We recall that if the Kolmogorov criterion fails, then the rate may well not be exponential (see Example 5 above).

The proof of this result is given in Section 6.

Remark 7 This result also sheds a different light on Theorem 2. Apparently, the counterexamples there depend on two things: on the one hand, the fact that there is no absolutely continuous component, which would give more independence, and – on the other hand – the special structure of the singular component making that part of the process well-approximable.

Remark 8 In the case when (ξ_j) is a standard normal i.i.d. sequence, $e^{\mathcal{C}(f)}$ has a spectral interpretation as the largest eigenvalue of the self-adjoint linear operator $R : L_2[-f, f] \mapsto L_2[-f, f]$ given by

$$[Rg](x) := \int_{-f}^f \phi(x-y)g(y)dy$$

where ϕ is the standard normal density. Namely, let $u(x, n)$ be the probability to stay in $[-f, f]$ for n steps of a random random walk with standard Gaussian steps starting at x . Formally, $u(x, 0) = 1$ for $x \in [-f, f]$. Then

$$u(n+1, x) = [Ru(n, \cdot)](x)$$

and by induction and the spectral theorem one obtains

$$u(n, x) = [R^n 1](x) = \sum_k \lambda_k^n \langle \psi_k, 1 \rangle \psi_k(x) \sim c(x) \lambda_1^n$$

where (λ_k, ψ_k) are pairs of (decreasing) eigenvalues and eigenfunctions of R . We refer to [4] for several variations of this approach that clearly does not seem to work beyond the case of independent sequences.

2 Proof of Theorem 1

2.1 Preliminaries

First, we shall make an extensive use of the recently proved Gaussian correlation inequality [17, 9]. It states that for any centered Gaussian measure μ on \mathbb{R}^d and any closed, convex, symmetric sets B_1, B_2 one has

$$\mu(B_1 \cap B_2) \geq \mu(B_1) \cdot \mu(B_2).$$

We shall use it in the form

$$\begin{aligned} & \mathbb{P}\{\max_{k \in A_1} |X_k| \leq \varepsilon_1; \max_{k \in A_2} |X_k| \leq \varepsilon_2\} \\ & \geq \mathbb{P}\{\max_{k \in A_1} |X_k| \leq \varepsilon_1\} \cdot \mathbb{P}\{\max_{k \in A_2} |X_k| \leq \varepsilon_2\}, \end{aligned} \quad (10)$$

for centered Gaussian vectors $X = (X_k)_{1 \leq k \leq d}$ and index sets $A_1, A_2 \subseteq \{1, \dots, d\}$, and $d \in \mathbb{N}$.

Second, we shall recall the extended Talagrand lower bound for small deviation probabilities, which will be used at various occasions. For this purpose, let $(X_t)_{t \in T}$ be a centered Gaussian process. We define the Dudley metric by

$$\rho(t, s) := \mathbb{E}[|X_t - X_s|^2]^{1/2}$$

and the corresponding covering numbers of T by

$$N_c(h) := \min\{n \mid \exists t_1, \dots, t_n \in T : \min_{i=1, \dots, n} \rho(t, t_i) \leq h \text{ for all } t \in T\}.$$

Then the extended Talagrand bound for small deviations that we shall use (Theorem 2 from [1], see the original Talagrand's version p. 257 in [10] and [20]) says that if for some function Ψ we have $N_c(h) \leq \Psi(h)$ and

$$\Psi(h/2) \leq C \Psi(h), \quad (11)$$

for some $C > 1$, then we have

$$\log \mathbb{P}\left\{\sup_{t, s \in T} |X_t - X_s| \leq c_0 h\right\} \geq -c \tilde{\Psi}(h), \quad (12)$$

with some numerical constant c_0 and constant $c > 0$ depending on C , and

$$\tilde{\Psi}(h) := \int_h^{\text{diam}(T, \rho)} \frac{\Psi(u)}{u} du.$$

Finally, let us introduce the regularly varying function

$$d(r) := r^{1/H} L(r)^{1/H}. \quad (13)$$

Then by using (5) we have as $r \rightarrow \infty$

$$\begin{aligned} d(r)^H \sqrt{\ell(d(r)^{-1})} &= r L(r) \sqrt{\ell(r^{-1/H} L(r)^{-1/H})} \\ &= r L(r) \tilde{\ell}(r L(r)) \sim r. \end{aligned} \quad (14)$$

2.2 Proof of the theorem

Let $M > 0$ be a large constant, and let $\varepsilon > 0$ be a small constant such that $0 < \varepsilon < \frac{1}{c_0}$, where c_0 is the numerical constant from extended Talagrand lower bound (12).

Set $\Delta := \Delta_N := \lfloor d(M^H f_N) \rfloor$, where the function d was defined in (13), and $A := \{j\Delta, 0 \leq j \leq \frac{N}{\Delta}\}$.

Note that by (14), we have $\Delta^H \ell(\Delta^{-1})^{1/2} \sim M^H f_N$. Let N be large enough such that

$$\Delta^H \ell(\Delta^{-1})^{1/2} \leq (1 + \varepsilon) M^H f_N.$$

Using the Gaussian correlation inequality (10), we obtain

$$\begin{aligned}
& \mathbb{P}\left\{\max_{1 \leq n \leq N} |S_n| \leq f_N\right\} \\
& \geq \mathbb{P}\left\{\max_{a \in A} |S_a| \leq c_0 \varepsilon f_N, \max_{a \in A} \max_{1 \leq n \leq \Delta} |S_{a+n} - S_a| \leq (1 - c_0 \varepsilon) f_N\right\} \\
& \geq \mathbb{P}\left\{\max_{a \in A} |S_a| \leq c_0 \varepsilon f_N\right\} \prod_{a \in A} \mathbb{P}\left\{\max_{1 \leq n \leq \Delta} |S_{a+n} - S_a| \leq (1 - c_0 \varepsilon) f_N\right\} \\
& \geq \mathbb{P}\left\{\max_{a \in A} |S_a| \leq c_0 \varepsilon f_N\right\} \mathbb{P}\left\{\max_{1 \leq n \leq \Delta} |S_n| \leq (1 - c_0 \varepsilon) f_N\right\}^{N/\Delta+1} \\
& =: \mathbb{P}_1 \mathbb{P}_2^{N/\Delta+1}.
\end{aligned}$$

For \mathbb{P}_2 , we use weak convergence, with M and ε fixed and N going to infinity and obtain:

$$\begin{aligned}
\mathbb{P}_2 &= \mathbb{P}\left\{\max_{1 \leq n \leq \Delta} \frac{|S_n|}{\Delta^H \ell(\Delta^{-1})^{1/2}} \leq \frac{(1 - c_0 \varepsilon) f_N}{\Delta^H \ell(\Delta^{-1})^{1/2}}\right\} \\
&\geq \mathbb{P}\left\{\max_{1 \leq n \leq \Delta} \frac{|S_n|}{\Delta^H \ell(\Delta^{-1})^{1/2}} \leq \frac{(1 - c_0 \varepsilon) f_N}{(1 + \varepsilon) M^H f_N}\right\} \\
&\geq \mathbb{P}\left\{\max_{1 \leq n \leq \Delta} \frac{|S_n|}{\Delta^H \ell(\Delta^{-1})^{1/2}} \leq \frac{1 - c_0 \varepsilon}{(1 + \varepsilon) M^H}\right\} \\
&\rightarrow \mathbb{P}\left\{\max_{0 \leq t \leq 1} |W^H(t)| \leq \frac{1 - c_0 \varepsilon}{(1 + \varepsilon) M^H}\right\}.
\end{aligned}$$

For every fixed $\varepsilon_1 > 0$ for M large enough by using small deviation asymptotics of FBM (1) we have

$$\mathbb{P}\left\{\max_{0 \leq t \leq 1} |W^H(t)| \leq \frac{1 - c_0 \varepsilon}{(1 + \varepsilon) M^H}\right\} \geq \exp\left\{-\kappa_H(1 + \varepsilon_1) \frac{M(1 + \varepsilon)^{1/H}}{(1 - c_0 \varepsilon)^{1/H}}\right\}.$$

Note that our theorem's assumption $\frac{f_N}{N^H \ell(1/N)^{1/2}} \rightarrow 0$ is equivalent to $N/\Delta \rightarrow \infty$. Indeed, as we see from (14), the function $d(\cdot)$ is an asymptotic inverse (see [19, Section 1.6]) to the function $g : d \mapsto d^H \sqrt{\ell(1/d)}$. Therefore,

$$\frac{M f_N}{N^H \ell(1/N)^{1/2}} = \frac{M f_N}{g(N)} \rightarrow 0 \quad \text{is equivalent to} \quad \frac{\Delta}{N} \sim \frac{d(M f_N)}{d(g(N))} \rightarrow 0.$$

So, since $N/\Delta \sim N M^{-1} [f_N L(f_N)]^{-1/H}$, it follows that for large N

$$\mathbb{P}_2^{N/\Delta+1} \geq \exp\left\{-\kappa_H(1 + 2\varepsilon_1)(1 + \varepsilon)^{1/H} \frac{N [f_N L(f_N)]^{-1/H}}{(1 - c_0 \varepsilon)^{1/H}}\right\}. \quad (15)$$

We continue with the evaluation of \mathbb{P}_1 by using the extended Talagrand inequality (12) as a tool. Let $N_c(\cdot)$ denote the covering numbers for the

process $\{S_a, a \in A\}$. Weak convergence yields (for large N , by using $f_N \rightarrow \infty$)

$$\mathbb{E} |S_a - S_b|^2 \leq C^2 |a - b|^{2H} \ell((a - b)^{-1}), \quad a, b \in A.$$

In the following we denote by C large constants, not depending on N , that may be different from line to line. It follows that

$$\begin{aligned} N_c(h) &\leq \Psi(h) := \min\{|A|, CNh^{-1/H}L(h)^{-1/H}\} \\ &\leq CN \min\{M^{-1}[f_N L(f_N)]^{-1/H}, h^{-1/H}L(h)^{-1/H}\} \\ &= \begin{cases} CNM^{-1}[f_N L(f_N)]^{-1/H}, & h < h_*, \\ CNh^{-1/H}L(h)^{-1/H}, & h \geq h_*, \end{cases} \end{aligned}$$

with $h_* \sim CM^H f_N$. Notice that the main assumption (11) of the extended Talagrand lower bound is verified because

$$\frac{\Psi(h/2)}{\Psi(h)} \leq \frac{(h/2)^{-1/H}L(h/2)^{-1/H}}{h^{-1/H}L(h)^{-1/H}} \leq C2^{1/H}, \quad \forall h > 0.$$

Letting

$$\tilde{\Psi}(r) := \int_r^\infty \frac{\Psi(h)}{h} dh$$

we have by (12)

$$\mathbb{P}_1 = \mathbb{P} \left\{ \max_{a \in A} |S_a| \leq c_0 \varepsilon f_N \right\} \geq \exp \left\{ -C \tilde{\Psi}(\varepsilon f_N) \right\}.$$

We finally get the key estimate for $\tilde{\Psi}(\varepsilon f_N)$. Namely,

$$\begin{aligned} \tilde{\Psi}(\varepsilon f_N) &\leq \left(\int_{\varepsilon f_N}^{h_*} + \int_{h_*}^\infty \right) \frac{\Psi(h)}{h} dh \\ &\leq CNM^{-1}[f_N L(f_N)]^{-1/H} \ln \left(\frac{h_*}{\varepsilon f_N} \right) + CNh_*^{-1/H}L(h_*)^{-1/H} \\ &\leq CNM^{-1}[f_N L(f_N)]^{-1/H} \ln \left(\frac{CM^H f_N}{\varepsilon f_N} \right) + CNM^{-1}[f_N L(f_N)]^{-1/H} \\ &= C \left(M^{-1} \ln \left(\frac{CM^H}{\varepsilon} \right) + M^{-1} \right) N [f_N L(f_N)]^{-1/H}. \end{aligned}$$

We conclude that

$$\mathbb{P}_1 \geq \exp \left\{ -C \left(M^{-1} \ln \left(\frac{CM^H}{\varepsilon} \right) + C M^{-1} \right) N [f_N L(f_N)]^{-1/H} \right\}. \quad (16)$$

By combining (15) and (16) we obtain for large N and M ,

$$\begin{aligned} &\frac{\ln \mathbb{P}\{\max_{1 \leq n \leq N} |S_n| \leq f_N\}}{N [f_N L(f_N)]^{-1/H}} \\ &\geq -\kappa_H(1 + 2\varepsilon_1)(1 + \varepsilon)^{1/H}(1 - c_0\varepsilon)^{-1/H} - C \left(M^{-1} \ln \left(\frac{CM^H}{\varepsilon} \right) + M^{-1} \right). \end{aligned}$$

By letting first $N \rightarrow \infty$ and then $M \rightarrow \infty$ with $\varepsilon, \varepsilon_1$ fixed, we have

$$\liminf_{N \rightarrow \infty} \frac{\ln \mathbb{P}\{\max_{1 \leq n \leq N} |S_n| \leq f_N\}}{N [f_N L(f_N)]^{-1/H}} \geq -\kappa_H (1 + 2\varepsilon_1)(1 + \varepsilon)(1 - c_0\varepsilon)^{-1/H}.$$

Then, letting $\varepsilon, \varepsilon_1 \searrow 0$, we obtain

$$\liminf_{N \rightarrow \infty} \frac{\ln \mathbb{P}\{\max_{1 \leq n \leq N} |S_n| \leq f_N\}}{N [f_N L(f_N)]^{-1/H}} \geq -\kappa_H,$$

as required.

3 Proof of Theorem 3

3.1 Preliminaries

First note that the lower bound in Theorem 3 follows from Theorem 1, since (8) implies (2) and thus (3). However, we shall still give an independent proof of the lower bound in Theorem 3 for the special case that μ is itself absolutely continuous with density p . This is because parts of this proof will be used in the proof of the *upper* bound in Theorem 3 and also in the proof of Theorem 2 (namely (19) and (21) as well as (22)).

First, it is useful to evaluate the variances of the partial sums

$$\begin{aligned} \mathbb{E} |S_n|^2 &= \int_{-\pi}^{\pi} |e^{inu} - 1|^2 \frac{p(u) du}{|1 - e^{iu}|^2} \\ &= n^{-1} \int_{-n\pi}^{n\pi} |e^{iv} - 1|^2 \frac{p(v/n) dv}{|1 - e^{iv/n}|^2} \\ &\sim \ell(1/n) m_H n^{-1} \int_{-\infty}^{\infty} |e^{iv} - 1|^2 (|v|/n)^{-1-2H} dv \\ &= \ell(1/n) n^{2H}. \end{aligned} \tag{17}$$

In particular, if we let $n \sim d(r)$, we get $\mathbb{E} |S_n|^2 \sim r^2$ by (14).

In the following, we split the spectral measure into three pieces by fixing a large $M > 0$ and restricting the spectral measure to the sets $\{|u| < \frac{1}{Md(f_N)}\}$, $\{\frac{1}{Md(f_N)} \leq |u| \leq \frac{M}{d(f_N)}\}$, and $\{|u| > \frac{M}{d(f_N)}\}$, respectively. The sequence (ξ_j) splits into the sum of three independent sequences $\xi^{(1)}, \xi^{(2)}, \xi^{(3)}$. The corresponding partial sums will be denoted $S_{z,n}$ with $z = 1, 2, 3$. From the small deviation viewpoint, $S_{2,n}$ is the main term, while two others are inessential remainders.

Let us finally mention that at various places below we will use the following form of Anderson's inequality. Let (ξ_j) and (ξ'_j) be stationary real centered Gaussian sequences with spectral measures μ and μ' , respectively,

and $\mu = \mu' + \nu$ with another (positive) measure ν . If (S_n) , (S'_n) are the partial sums corresponding to (ξ_j) , (ξ'_j) , then for any $f > 0$ it is true that

$$\mathbb{P}\{\max_{1 \leq n \leq N} |S_n| \leq f\} \leq \mathbb{P}\{\max_{1 \leq n \leq N} |S'_n| \leq f\}. \quad (18)$$

Indeed, one can construct a probability space with an independent stationary real centered Gaussian sequences (ξ''_j) and (η_j) such that (ξ''_j) is equidistributed with (ξ'_j) and (η_j) has the spectral measure ν . Then the sum $(\xi''_j + \eta_j)$ has the spectral measure $\mu' + \nu = \mu$, i.e. it is equidistributed with (ξ_j) . If (S''_n) and (T_n) are the partial sums corresponding to (ξ''_j) and (η_j) , respectively, we obtain from Anderson's inequality [13, p. 135] that

$$\begin{aligned} \mathbb{P}\{\max_{1 \leq n \leq N} |S_n| \leq f\} &= \mathbb{P}\{\max_{1 \leq n \leq N} |S''_n + T_n| \leq f\} \\ &= \mathbb{E}_\eta[\mathbb{P}_{\xi''}\{\max_{1 \leq n \leq N} |S''_n + T_n| \leq f\}] \\ &\leq \mathbb{P}\{\max_{1 \leq n \leq N} |S''_n| \leq f\} = \mathbb{P}\{\max_{1 \leq n \leq N} |S'_n| \leq f\}, \end{aligned}$$

and (18) is justified.

3.2 Proof of the lower bound

We wish to prove that for any fixed $\delta \in (0, \frac{1}{2})$

$$\ln \mathbb{P}\{\max_{1 \leq n \leq N} |S_{1,n}| \leq \delta f_N\} \geq -\delta_M [L(f_N) f_N]^{-1/H} N, \quad (19)$$

$$\ln \mathbb{P}\{\max_{1 \leq n \leq N} |S_{2,n}| \leq (1 - 2\delta) f_N\} \geq -\kappa_H [(1 - 3\delta) L(f_N) f_N]^{-1/H} N, \quad (20)$$

$$\ln \mathbb{P}\{\max_{1 \leq n \leq N} |S_{3,n}| \leq \delta f_N\} \geq -\delta_M [L(f_N) f_N]^{-1/H} N, \quad (21)$$

where δ_M goes to zero when M tends to infinity. It follows from the independence of the $S_{z,n}$, $z = 1, 2, 3$, that

$$\begin{aligned} &\ln \mathbb{P}\{\max_{1 \leq n \leq N} |S_n| \leq f_N\} \\ &= \ln \mathbb{P}\{\max_{1 \leq n \leq N} |S_{1,n} + S_{2,n} + S_{3,n}| \leq f_N\} \\ &\geq \ln \mathbb{P}\{\max_{1 \leq n \leq N} |S_{1,n}| \leq \delta f_N, \max_{0 \leq n \leq N} |S_{2,n}| \leq (1 - 2\delta) f_N, \max_{1 \leq n \leq N} |S_{3,n}| \leq \delta f_N\} \\ &= \ln \mathbb{P}\{\max_{1 \leq n \leq N} |S_{1,n}| \leq \delta f_N\} + \ln \mathbb{P}\{\max_{1 \leq n \leq N} |S_{2,n}| \leq (1 - 2\delta) f_N\} \\ &\quad + \ln \mathbb{P}\{\max_{1 \leq n \leq N} |S_{3,n}| \leq \delta f_N\} \\ &\geq -\left[2\delta_M + \kappa_H (1 - 3\delta)^{-1/H}\right] [L(f_N) f_N]^{-1/H} N, \end{aligned}$$

which provides us with the correct lower bound in Theorem 3.

3.2.1 Lower frequencies

First, we aim at showing (19). We shall use the extended Talagrand bound (12) for the small deviations of $(S_{1,n})_{n \leq N}$. For this purpose, consider the related Dudley metric: for $n, m \leq N$,

$$\begin{aligned}
\mathbb{E} |S_{1,n} - S_{1,m}|^2 &= \mathbb{E} |S_{1,|n-m|}|^2 \\
&= \int_{|u| < \frac{1}{Md(f_N)}} \left| \frac{e^{i|n-m|u} - 1}{e^{iu} - 1} \right|^2 p(u) du \\
&\leq C \int_{|u| < \frac{1}{Md(f_N)}} \frac{|n-m|^2 u^2}{|e^{iu} - 1|^2} |u|^{1-2H} \ell(u) du \\
&\leq C |n-m|^2 \int_{|u| \leq 1/(Md(f_N))} |u|^{1-2H} \ell(u) du \\
&\leq C |n-m|^2 (Md(f_N))^{2H-2} \ell(d(f_N)^{-1}).
\end{aligned}$$

where C is a constant that does not depend on N (and that may change from line to line). This shows that

$$\begin{aligned}
(\mathbb{E} |S_{1,n} - S_{1,m}|^2)^{1/2} &\leq C |n-m| M^{H-1} d(f_N)^{-1} d(f_N)^H \ell(d(f_N)^{-1})^{1/2} \\
&\leq C |n-m| M^{H-1} d(f_N)^{-1} f_N,
\end{aligned}$$

where the last step is due to (14). From this bound for the Dudley metric one obtains for the covering numbers related to the process $S_{1,n}$:

$$N_c(\varepsilon) \leq C M^{H-1} d(f_N)^{-1} f_N N \varepsilon^{-1}, \quad \varepsilon > 0.$$

This shows, using (12) that

$$\begin{aligned}
\ln \mathbb{P}\{\max_{1 \leq n \leq N} |S_{1,n}| \leq \delta f_N\} &\geq -C M^{H-1} d(f_N)^{-1} f_N N \cdot (\delta f_N)^{-1} \\
&= -C \delta^{-1} M^{H-1} d(f_N)^{-1} N \\
&= -C \delta^{-1} M^{H-1} [L(f_N) f_N]^{-1/H} N,
\end{aligned}$$

where $\delta_M := C \delta^{-1} M^{H-1}$ tends to zero as $M \rightarrow \infty$, as required by (19).

3.2.2 Main frequencies

Using the uniform convergence property of slowly varying functions, cf. [19, Theorem 1.1], in the frequency zone $\{\frac{1}{Md(f_N)} \leq |u| \leq \frac{M}{d(f_N)}\}$ we have

$$\begin{aligned}
p(u) &\sim m_H \ell(u) |u|^{1-2H} \sim m_H \ell(d(f_N)^{-1}) |u|^{1-2H} \\
&\sim m_H f_N^2 d(f_N)^{-2H} |u|^{1-2H} \\
&= m_H L(f_N)^{-2} |u|^{1-2H}.
\end{aligned} \tag{22}$$

The latter expression coincides with the asymptotics of p_{FGN} up to the constant factor $L(f_N)^{-2}$. Therefore, for any $\delta_1 > 0$ for large N we have

$$p(u) \leq L(f_N)^{-2}(1 + \delta_1)^2 p_{\text{FGN}}(u), \quad u \in [-\pi, \pi].$$

By Anderson's inequality, cf. (18), for any $\varepsilon > 0$ we have

$$\begin{aligned} \mathbb{P}\{\max_{1 \leq n \leq N} |S_{2,n}| \leq \varepsilon\} &\geq \mathbb{P}\{\max_{1 \leq n \leq N} |W^H(n)| \leq L(f_N)(1 + \delta_1)^{-1} \varepsilon\} \\ &\geq \mathbb{P}\{\sup_{0 \leq t \leq N} |W^H(t)| \leq L(f_N)(1 + \delta_1)^{-1} \varepsilon\}. \end{aligned}$$

By letting here $\varepsilon = (1 - 2\delta)f_N$ and using the small deviation asymptotics for W_H from (1), we obtain

$$\begin{aligned} \ln \mathbb{P}\{\max_{1 \leq n \leq N} |S_{2,n}| \leq (1 - 2\delta)f_N\} \\ \geq -(1 + o(1))\kappa_H [L(f_N)(1 + \delta_1)^{-1}(1 - 2\delta)f_N]^{-1/H} N. \end{aligned}$$

By letting $\delta_1 \rightarrow 0$ this simplifies to

$$\ln \mathbb{P}\{\max_{1 \leq n \leq N} |S_{2,n}| \leq (1 - 2\delta)f_N\} \geq -\kappa_H [L(f_N)(1 - 3\delta)f_N]^{-1/H} N.$$

as announced in (20).

3.2.3 Higher frequencies

One of the main points of the evaluation here is the uniform bound for variances. We have, by using (14) at the end,

$$\begin{aligned} \mathbb{E}|S_{3,n}|^2 &= \int_{|u| > \frac{M}{d(f_N)}} |e^{inu} - 1|^2 \frac{p(u)du}{|1 - e^{iu}|^2} \\ &\leq 4 \int_{|u| > \frac{M}{d(f_N)}} \frac{p(u)du}{|1 - e^{iu}|^2} \\ &\sim 4 \int_{|u| > \frac{M}{d(f_N)}} \frac{\ell(u)du}{|u|^{1+2H}} \\ &\leq C M^{-2H} \ell\left(\frac{M}{d(f_N)}\right) d(f_N)^{2H} \\ &\sim C M^{-2H} \ell(d(f_N)^{-1}) d(f_N)^{2H} \\ &\sim C M^{-2H} f_N^2. \end{aligned} \tag{23}$$

We will show that for every $0 \leq j < N/d(f_N)$

$$\ln \mathbb{P}\{\max_{jd(f_N) < n \leq (j+1)d(f_N)} |S_{3,n}| \leq \delta f_N\} \geq -\delta_M. \tag{24}$$

Once this is done, by the correlation inequality (10) and the definition of d , it follows that

$$\begin{aligned} \ln \mathbb{P}\{\max_{1 \leq n \leq N} |S_{3,n}| \leq \delta f_N\} &\geq \sum_{j \leq \frac{N}{d(f_N)}} \ln \mathbb{P}\{\max_{jd(f_N) \leq k \leq (j+1)d(f_N)} |S_{3,n}| \leq \delta f_N\} \\ &\geq -\frac{N}{d(f_N)} \delta_M = -\delta_M N [f_N L(f_N)]^{-1/H}, \end{aligned}$$

as required in (21). Next, for proving (24), we use the correlation inequality (10) and separate the initial points:

$$\begin{aligned} &\mathbb{P}\{\max_{jd(f_N) \leq n \leq (j+1)d(f_N)} |S_{3,n}| \leq \delta f_N\} \\ &\geq \mathbb{P}\{|S_{3,jd(f_N)}| \leq \frac{\delta f_N}{2}\} \cdot \mathbb{P}\{\max_{jd(f_N) \leq n \leq (j+1)d(f_N)} |S_{3,n} - S_{3,jd(f_N)}| \leq \frac{\delta f_N}{2}\} \\ &= \mathbb{P}\{|S_{3,jd(f_N)}| \leq \frac{\delta f_N}{2}\} \cdot \mathbb{P}\{\max_{1 \leq n \leq d(f_N)} |S_{3,n}| \leq \frac{\delta f_N}{2}\}. \end{aligned}$$

For the first factor, we use the variance bound (23) and obtain that (for a standard normal \mathcal{N})

$$\mathbb{P}\{|S_{3,jd(f_N)}| \leq \delta f_N/2\} \geq \mathbb{P}\{|\mathcal{N}| \leq C^{-1/2} M^H \delta/2\},$$

which is close to one for large M .

For the second factor let

$$\mathcal{D} := \int_0^\infty \sqrt{\ln N_c(r)} \, dr$$

be the Dudley integral of covering numbers $N_c(r)$ corresponding to the process $(S_{3,n})_{1 \leq n \leq d(f_N)}$. Denote m and E the median and the expectation of $\max_{1 \leq n \leq d(f_N)} S_{3,n}$. It is known from the general Gaussian theory that $m \leq E \leq C_D \mathcal{D}$ with $C_D = 4\sqrt{2}$, cf. [13, Section 14, Theorem 1]. Let also

$$\sigma^2 := \max_{1 \leq n \leq d(f_N)} \mathbb{E} |S_{3,n}|^2.$$

We know from (23) that $\sigma^2 \leq C M^{-2H} f_N^2$. Then for any $r > C_D \mathcal{D}$ by the Gaussian concentration inequality, cf. [13, Section 12, Theorem 2],

$$\begin{aligned} \mathbb{P}\{\max_{1 \leq n \leq d(f_N)} |S_{3,n}| \geq r\} &\leq 2 \mathbb{P}\{\max_{1 \leq n \leq d(f_N)} S_{3,n} \geq r\} \\ &= 2 \mathbb{P}\{\max_{1 \leq n \leq d(f_N)} S_{3,n} - m \geq r - m\} \\ &\leq 2 \mathbb{P}\{\max_{1 \leq n \leq d(f_N)} S_{3,n} - m \geq r - E\} \\ &\leq 2 \mathbb{P}\{\max_{1 \leq n \leq d(f_N)} S_{3,n} - m \geq r - C_D \mathcal{D}\} \\ &\leq 2 \mathbb{P}\{\mathcal{N} \geq (r - C_D \mathcal{D})/\sigma\}. \end{aligned}$$

We apply this bound with $r = \delta f_N/2$. If we are able to prove that

$$\mathcal{D} \leq h_M f_N \quad (25)$$

with arbitrarily small h_M for large M , then we get for large M

$$\begin{aligned} \mathbb{P}\left\{\max_{1 \leq n \leq d(f_N)} |S_{3,n}| \geq \delta f_N/2\right\} &\leq 2 \mathbb{P}\{\mathcal{N} \geq (\delta/2 - C_D h_M)/\sqrt{C} M^{-H}\} \\ &\leq 2 \mathbb{P}\{\mathcal{N} \geq \delta M^H/4\sqrt{C}\} \\ &= \mathbb{P}\{|\mathcal{N}| \geq \delta M^H/4\sqrt{C}\}, \end{aligned}$$

or equivalently

$$\mathbb{P}\left\{\max_{1 \leq n \leq d(f_N)} |S_{3,n}| \leq \delta f_N/2\right\} \geq \mathbb{P}\{|\mathcal{N}| \leq \delta M^H/4\sqrt{C}\},$$

which is close to 1, as required for (24).

It remains to justify (25). First of all, notice that for all integers n_1, n_2 it is true that

$$\begin{aligned} \mathbb{E} |S_{3,n_1} - S_{3,n_2}|^2 &\leq \mathbb{E} |S_{n_1} - S_{n_2}|^2 \\ &= \mathbb{E} |S_{|n_1 - n_2|}|^2 \sim \ell \left(\frac{1}{|n_1 - n_2|} \right) |n_1 - n_2|^{2H}, \end{aligned}$$

having used (17). Hence, we have a uniform bound

$$\mathbb{E} |S_{3,n_1} - S_{3,n_2}|^2 \leq A_1 \ell \left(\frac{1}{|n_1 - n_2|} \right) |n_1 - n_2|^{2H}.$$

We will also use the bounds

$$\ell \left(\frac{1}{n} \right) n^{2H} \leq A_2 \ell \left(\frac{1}{d} \right) d^{2H} \quad \forall n \leq d,$$

and, by using (14)

$$\ell \left(\frac{1}{d(r)} \right) d(r)^{2H} \leq A_3 r^2.$$

It follows now that $|n_1 - n_2| \leq d(r)$ yields

$$\mathbb{E} |S_{3,n_1} - S_{3,n_2}|^2 \leq A_1 A_2 \ell \left(\frac{1}{d(r)} \right) d(r)^{2H} \leq A_1 A_2 A_3 r^2 =: (Ar)^2.$$

For the covering numbers of the process $(S_{3,n})_{1 \leq n \leq d(f_N)}$ this means

$$N_c(Ar) \leq \frac{d(f_N)}{d(r)}, \quad r \geq r_0,$$

where r_0 is the point where the function d starts to be defined. Further, trivially

$$N_c(Ar) \leq d(f_N) \quad r > 0.$$

We shall use the second estimate for $r \in [0, f_N / \ln f_N]$ and the first estimate for $r > f_N / \ln f_N$. Namely,

$$\begin{aligned}
\mathcal{D} &\leq \int_0^\sigma \sqrt{\ln N_c(\rho)} d\rho \\
&= A \int_0^{\sigma/A} \sqrt{\ln N_c(Ar)} dr \\
&\leq A \int_0^{f_N / \ln f_N} \sqrt{\ln d(f_N)} dr + A \int_{f_N / \ln f_N}^{\sigma/A} \sqrt{\ln \frac{d(f_N)}{d(r)}} dr \\
&\leq C f_N (\ln f_N)^{-1/2} + A f_N \int_{1/\ln f_N}^{\sqrt{C} M^{-H}/A} \sqrt{\ln \frac{d(f_N)}{d(f_N v)}} dv,
\end{aligned}$$

where we used that d is a regularly varying function (so that $d(f_N) \leq f_N^c$ for large N) and that $\sigma \leq \sqrt{C} M^{-H}$ from (23). The first term already satisfies the claim (25), so we shall look at the second term now.

Since the function $d(\cdot)$ is $\frac{1}{H}$ -regularly varying, we have, for $N \rightarrow \infty$ (and so $f_N v \rightarrow \infty$ on the range for v considered here):

$$\frac{d(f_N)}{d(f_N v)} = \frac{d(f_N v \cdot v^{-1})}{d(f_N v)} \rightarrow (v^{-1})^{1/H}.$$

and so the fraction is bounded above by $v^{-2/H}$, say, for large N . Hence, for large N

$$\mathcal{D} \leq C f_N M^{-H} + A f_N \int_0^{\sqrt{C} M^{-H}/A} \frac{1}{\sqrt{2} |\ln v|^{1/H}} dv =: h_M f_N,$$

as required in (25).

3.3 Proof of the upper bound

3.3.1 Fractional Gaussian noise

First note that for fractional Brownian motion, we can estimate the continuous time maximum by the discrete time maximum as follows: Fix $h > 0$.

Then using the correlation inequality (10)

$$\begin{aligned}
& \mathbb{P}\{\max_{t \in [0, N]} |W_t^H| \leq (1+h)f_N\} \\
& \geq \mathbb{P}\{\max_{n=1, \dots, N} |W_n^H| \leq f_N, \max_{n=1, \dots, N} \max_{t \in [0, 1]} |W_{n-1+t}^H - W_{n-1}^H| \leq hf_N\} \\
& \geq \mathbb{P}\{\max_{n=1, \dots, N} |W_n^H| \leq f_N\} \cdot \mathbb{P}\{\max_{n=1, \dots, N} \max_{t \in [0, 1]} |W_{n-1+t}^H - W_{n-1}^H| \leq hf_N\} \\
& \geq \mathbb{P}\{\max_{n=1, \dots, N} |W_n^H| \leq f_N\} \cdot \mathbb{P}\{\max_{t \in [0, 1]} |W_t^H| \leq hf_N\}^N \\
& = \mathbb{P}\{\max_{n=1, \dots, N} |W_n^H| \leq f_N\} \cdot \exp(N \ln(1 - \mathbb{P}\{\max_{t \in [0, 1]} |W_t^H| > hf_N\})) \\
& \geq \mathbb{P}\{\max_{n=1, \dots, N} |W_n^H| \leq f_N\} \cdot \exp(-2N \mathbb{P}\{\max_{t \in [0, 1]} |W_t^H| > hf_N\}) \\
& \geq \mathbb{P}\{\max_{n=1, \dots, N} |W_n^H| \leq f_N\} \cdot \exp(-2Ne^{-h^2 f_N^2/2}) \\
& \geq \mathbb{P}\{\max_{n=1, \dots, N} |W_n^H| \leq f_N\} \cdot \exp(-3Nh^{-2/H} f_N^{-2/H}),
\end{aligned}$$

for N large enough. This shows, using the small deviation asymptotics of FBM (1),

$$\begin{aligned}
& \ln \mathbb{P}\{\max_{n=1, \dots, N} |W_n^H| \leq f_N\} \\
& \leq -\kappa_H (1+h)^{-1/H} f_N^{-1/H} N \cdot (1+o(1)) + 3h^{-2/H} f_N^{-2/H} N \\
& \leq -\kappa_H (1+h)^{-1/H} f_N^{-1/H} N \cdot (1+o(1)).
\end{aligned}$$

Letting $h \rightarrow 0$, this proves that

$$\ln \mathbb{P}\{\max_{n=1, \dots, N} |W_n^H| \leq f_N\} \leq -\kappa_H f_N^{-1/H} N \cdot (1+o(1)). \quad (26)$$

3.3.2 Proof of the general upper bound

The first step is to cut off the part of the spectral measure that belongs to the singular component. Let (S_n^μ) and (S_n^p) be the partial sums of correlated stationary Gaussian random variables with spectral measures $\mu(du) = p(u)du + \mu_s(du)$ and $p(u)du$, respectively. Then, by Anderson's inequality, cf. (18),

$$\mathbb{P}\{\max_{n=1, \dots, N} |S_n^\mu| \leq f_N\} \leq \mathbb{P}\{\max_{n=1, \dots, N} |S_n^p| \leq f_N\}.$$

Therefore, we can assume w.l.o.g. that μ is absolutely continuous and has a spectral density p satisfying (8).

Fix $M > 0$ and $\delta > 0$. We saw in (22) that on the frequency zone $\{\frac{1}{Md(f_N)} \leq |u| \leq \frac{M}{d(f_N)}\}$ we have

$$p(u) \sim m_H L(f_N)^{-2} |u|^{1-2H},$$

which up to the factor $L(f_N)^{-2}$ is the behavior of fractional Gaussian noise. Therefore, for large N , we have

$$p(u) \geq (1 - \delta)^2 L(f_N)^{-2} p_{\text{FGN}}(u), \quad \frac{1}{Md(f_N)} \leq |u| \leq \frac{M}{d(f_N)}.$$

Let us denote by $S_{2,n}$ the partial sums of correlated stationary Gaussian random variables with spectral measure $p(u) \mathbb{1}_{\{\frac{1}{Md(f_N)} \leq |u| \leq \frac{M}{d(f_N)}\}}$. Further, let $W^{H,1}$ and $W^{H,2}$ represent the processes related to the spectral densities $p_{\text{FGN}}(u) \mathbb{1}_{\{\frac{1}{Md(f_N)} \leq |u| \leq \frac{M}{d(f_N)}\}}^c$ and $p_{\text{FGN}}(u) \mathbb{1}_{\{\frac{1}{Md(f_N)} \leq |u| \leq \frac{M}{d(f_N)}\}}$, respectively.

Then by using Anderson's inequality (cf. (18)) twice we get:

$$\begin{aligned} & \mathbb{P}\{\max_{n=1,\dots,N} |S_n| \leq f_N\} \\ & \leq \mathbb{P}\{\max_{n=1,\dots,N} |S_{2,n}| \leq f_N\} \\ & \leq \mathbb{P}\{\max_{n=1,\dots,N} |(1 - \delta)L(f_N)^{-1}W_n^{H,2}| \leq f_N\} \\ & = \mathbb{P}\{\max_{n=1,\dots,N} |W_n^{H,2}| \leq (1 - \delta)^{-1}L(f_N)f_N\} \\ & \leq \frac{\mathbb{P}\{\max_{n=1,\dots,N} |W_n^{H,1} + W_n^{H,2}| \leq (1 + \delta)(1 - \delta)^{-1}L(f_N)f_N\}}{\mathbb{P}\{\max_{n=1,\dots,N} |W_n^{H,1}| \leq \delta(1 - \delta)^{-1}L(f_N)f_N\}}, \end{aligned}$$

where we used the correlation inequality (10) in the last step.

The first term, by (26) is upper bounded by

$$\exp(-\kappa_H[(1 + \delta)(1 - \delta)^{-1}L(f_N)f_N]^{-1/H}N \cdot (1 + o(1))),$$

so that if it is true that

$$\begin{aligned} & \ln \mathbb{P}\{\max_{n=1,\dots,N} |W_n^{H,1}| \leq \delta(1 - \delta)^{-1}L(f_N)f_N\} \\ & \geq -\delta_M[L(f_N)f_N]^{-1/H}N(1 + o(1)), \end{aligned} \tag{27}$$

with $\delta_M \rightarrow 0$ as $M \rightarrow \infty$, we are done with the proof of the upper bound for S_n . However, note that (19) and (21) applied to $W^{H,1}$ imply (27).

4 Proof of Theorem 2

4.1 Preliminaries

Let us recall the spectral point of view. Recall that FBM is a process with stationary increments that can be written as a white noise integral

$$W^H(t) = \int_{\mathbb{R}} (e^{itu} - 1) \mathcal{W}(du)$$

where the control measure of the white noise \mathcal{W} is $\mu(du) = \frac{m_H du}{|u|^{2H+1}}$ and $m_H = \frac{\Gamma(2H+1) \sin(\pi H)}{2\pi}$ with $0 < H < 1$. The discrete time fractional Gaussian noise is a stationary sequence $\xi_j^{\text{FGN}} := W^H(j) - W^H(j-1)$. We have

$$\xi_j^{\text{FGN}} = \int_{\mathbb{R}} e^{iju} (1 - e^{-iu}) \mathcal{W}(du), \quad j \in \mathbb{N}.$$

Hence the spectral measure of (ξ_j^{FGN}) on $[-\pi, \pi)$, which we denote by ν_H , is the projection of the measure $\frac{m_H |1 - e^{-iu}|^2 du}{|u|^{2H+1}}$ by the mapping

$$x \mapsto 2\pi \left\{ \frac{x}{2\pi} \right\}, \quad x \in \mathbb{R},$$

where $\{\cdot\}$ denotes the fractional part of a real number.

The measure ν_H has a density p_{FGN} with singularity

$$p_{\text{FGN}}(u) \sim m_H |u|^{1-2H}, \quad u \rightarrow 0.$$

We shall construct a stationary Gaussian sequence with spectral measure that is a “perturbation” (to be defined precisely in the next subsection) of the spectral measure ν_H of fractional Gaussian noise. We shall see that this sequence (in fact any perturbation of ν_H) satisfies (2) and thus (3), see Proposition 12. On the other hand, we show that (7) can be made true, see (35).

Section 4 is structured as follows: in Subsection 4.2, we define what we mean by a perturbation of ν_H and show that any perturbation satisfies (2). In Subsection 4.3, we construct a concrete perturbation of ν_H , while Subsection 4.4 shows (7) for the sequence arising from that concrete construction.

4.2 Spectral measure perturbation

Along with ν_H introduce a measure $\tilde{\nu}_H(du) = \frac{\nu_H(du)}{|e^{iu} - 1|^2}$. Clearly, it has a density $\tilde{p}_H(u) = \frac{p_H(u)}{|e^{iu} - 1|^2} \sim m_H |u|^{-1-2H}$, as $u \rightarrow \infty$. Accordingly, we have

$$\tilde{\nu}_H[h, \pi] \sim \frac{m_H}{2H} h^{-2H}, \quad h \rightarrow 0.$$

We introduce a class of perturbations of ν_H and show that the same asymptotics holds for every measure of this class.

Defintion 9 *A symmetric measure Γ on $[-\pi, \pi]$ is called a perturbation of ν_H , if there exists a sequence $u_n \searrow 0$ such that $u_n/u_{n+1} \rightarrow 1$ and $\Gamma[u_n, \pi] = \nu_H[u_n, \pi]$.*

This simply means that Γ is obtained from ν_H by redistribution of the measure within the intervals $[u_{n+1}, u_n]$. We stress that Γ need not at all be absolutely continuous. On the contrary, a typical perturbation we will use is a partial discretization of ν_H .

As before, we denote $\tilde{\Gamma}(du) = \frac{\Gamma(du)}{|e^{iu} - 1|^2}$.

Lemma 10 *Let Γ be a perturbation of ν_H . Then*

$$\tilde{\Gamma}[h, \pi] \sim \frac{m_H}{2H} h^{-2H}, \quad h \rightarrow 0.$$

Proof: Let

$$\theta_n = \frac{\max_{u \in [u_{n+1}, u_n]} |e^{iu} - 1|}{\min_{u \in [u_{n+1}, u_n]} |e^{iu} - 1|}.$$

We clearly have $\theta_n \rightarrow 1$. For each n we have the bound

$$\frac{\tilde{\Gamma}[u_{n+1}, u_n]}{\tilde{\nu}_H[u_{n+1}, u_n]} \leq \frac{\Gamma[u_{n+1}, u_n]}{\nu_H[u_{n+1}, u_n]} \theta_n^2 = \theta_n^2,$$

and similarly

$$\frac{\tilde{\Gamma}[u_{n+1}, u_n]}{\tilde{\nu}_H[u_{n+1}, u_n]} \geq \theta_n^{-2}.$$

Therefore,

$$\frac{\tilde{\Gamma}[u_{n+1}, u_n]}{\tilde{\nu}_H[u_{n+1}, u_n]} \rightarrow 1,$$

and so

$$\tilde{\Gamma}[u_n, \pi] \sim \tilde{\nu}_H[u_n, \pi] \sim \frac{m_H}{2H} u_n^{-2H}.$$

Notice also that

$$\frac{\tilde{\nu}_H[u_{n+1}, \pi]}{\tilde{\nu}_H[u_n, \pi]} \rightarrow 1.$$

Finally, for $u \in [u_{n+1}, u_n]$ it is true that

$$\begin{aligned} \tilde{\Gamma}[u, \pi] &\leq \tilde{\Gamma}[u_{n+1}, \pi] \\ &= \frac{\tilde{\Gamma}[u_{n+1}, \pi]}{\tilde{\nu}_H[u_{n+1}, \pi]} \frac{\tilde{\nu}_H[u_{n+1}, \pi]}{\tilde{\nu}_H[u_n, \pi]} \frac{\tilde{\nu}_H[u_n, \pi]}{\tilde{\nu}_H[u, \pi]} \tilde{\nu}_H[u, \pi] \\ &\leq (1 + o(1)) \frac{m_H}{2H} u^{-2H}. \end{aligned}$$

The lower bound follows in the same way. □

Corollary 11 *Let Γ be a perturbation of ν_H . Then*

$$\int_{[-h, h]} u^2 \tilde{\Gamma}(du) \leq C h^{2-2H}, \quad 0 < h < \pi. \quad (28)$$

Proposition 12 *Let Γ be a perturbation of ν_H . Then for a stationary sequence with spectral measure Γ it is true that*

$$\mathbb{E} |S_n|^2 \sim n^{2H}, \quad \text{as } n \rightarrow \infty.$$

Proof: The spectral representation yields

$$\begin{aligned} \mathbb{E} |S_n|^2 &= \int_{-\pi}^{\pi} |e^{inu} - 1|^2 \frac{\Gamma(du)}{|e^{iu} - 1|^2} \\ &= \int_{-\pi}^{\pi} |e^{inu} - 1|^2 \tilde{\Gamma}(du) = 4 \int_0^{\pi} (1 - \cos(nu)) \tilde{\Gamma}(du). \end{aligned}$$

Let denote $F(u) = \tilde{\Gamma}[u, \pi]$. Integrating by parts implies

$$\int_0^{\pi} (1 - \cos(nu)) \tilde{\Gamma}(du) = n \int_0^{\pi} \sin(nu) F(u) du = \int_0^{n\pi} \sin(v) F(v/n) dv.$$

Let us fix a large positive *odd* integer V . Since $F(\cdot)$ is a decreasing function, for $n \geq V$ we have

$$\int_0^{n\pi} \sin(v) F(v/n) dv \leq \int_0^{V\pi} \sin(v) F(v/n) dv.$$

Furthermore, for any $\varepsilon > 0$ and all large n Lemma 10 yields

$$\int_0^{V\pi} \sin(v) F(v/n) dv = \frac{m_H}{2H} \int_0^{V\pi} \sin(v) (v/n)^{-2H} (1 + \vartheta_n(v)\varepsilon) dv$$

with $|\vartheta_n(v)| \leq 1$. It follows that

$$\int_0^{V\pi} \sin(v) F(v/n) dv \leq \frac{m_H}{2H} n^{2H} \left[\int_0^{V\pi} \sin(v) v^{-2H} dv + \varepsilon \int_0^{V\pi} |\sin(v)| v^{-2H} dv \right].$$

Hence,

$$\limsup_{n \rightarrow \infty} \frac{\mathbb{E} |S_n|^2}{n^{2H}} \leq \frac{2m_H}{H} \left[\int_0^{V\pi} \sin(v) v^{-2H} dv + \varepsilon \int_0^{V\pi} |\sin(v)| v^{-2H} dv \right].$$

By letting $\varepsilon \rightarrow 0$, then $V \rightarrow \infty$, we obtain

$$\limsup_{n \rightarrow \infty} \frac{\mathbb{E} |S_n|^2}{n^{2H}} \leq \frac{2m_H}{H} \int_0^{\infty} \sin(v) v^{-2H} dv.$$

The converse estimate

$$\liminf_{n \rightarrow \infty} \frac{\mathbb{E} |S_n|^2}{n^{2H}} \geq \frac{2m_H}{H} \int_0^{\infty} \sin(v) v^{-2H} dv$$

is obtained by the same way using large *even* V 's. We conclude that

$$\mathbb{E} |S_n|^2 \sim \left[\frac{2m_H}{H} \int_0^\infty \sin v \, v^{-2H} dv \right] n^{2H} = n^{2H}.$$

The constant in brackets is equal to 1 by the definition of m_H and by the well-known formula for the integral, see [7, Formula 858.811] (notice by the way that the integral is *absolutely* convergent only for $H > 1/2$, otherwise it is understood as the principal value). This constant also *must* be equal to 1, since the formula holds also for the non-perturbed case $\Gamma = \nu_H$ where we have the exact equality $\mathbb{E} |S_n|^2 = n^{2H}$. \square

4.3 Construction of the perturbed spectral measure

We first choose two sequences of positive reals $M_j \nearrow \infty$, and positive integers $q_j \rightarrow \infty$ such that

$$\lim_{j \rightarrow \infty} \frac{M_j^2}{q_j} = 0. \quad (29)$$

Denote $d(r) = r^{1/H}$. In our evaluations we strongly follow the proof of the lower bound in our spectral result in Section 3. For every N we may split the spectral domain into three parts $\{|u| \leq \frac{1}{Md(f_N)}\}$, $\{\frac{1}{Md(f_N)} \leq |u| \leq \frac{M}{d(f_N)}\}$, and $\{|u| \geq \frac{M}{d(f_N)}\}$, respectively. The sequence ξ splits into the sum of three independent sequences $\xi^{(1)}, \xi^{(2)}, \xi^{(3)}$. The corresponding partial sums will be denoted $S_{z,n}$ with $z = 1, 2, 3$.

We know that from the small deviation viewpoint, $S_{2,n}$ is the main term while two others are inessential remainders. Therefore, the main attention should be paid to the central part of the spectrum.

Let us choose a subsequence N_j increasing to infinity so quickly that the corresponding central domains do not overlap, i.e. $\frac{1}{M_j d(N_j)} > \frac{M_{j+1}}{d(N_{j+1})}$, which can be rewritten as

$$d(N_{j+1}) > M_{j+1} M_j d(N_j). \quad (30)$$

We also need another growth condition

$$\lim_{j \rightarrow \infty} \frac{|\ln P(M_j, q_j)|}{N_j f_{N_j}^{-1/H}} = 0, \quad (31)$$

where $P(M, q)$ is a function explicitly defined below in (36). At the moment we do not care about its form but only stress that the construction of such (N_j) is possible due to the assumption $N f_N^{-1/H} \rightarrow \infty$.

Once all sequences are constructed, we build the perturbation Γ of the measure ν_H . For each j we discretize the measure ν_H on the zone $\{\frac{1}{M_j d(f_{N_j})} \leq |u| \leq \frac{M_j}{d(f_{N_j})}\}$, by spreading over it uniformly q_j points

$$t_{j,k} := \frac{1}{M_j d(f_{N_j})} + \frac{k}{q_j} \left(M_j - \frac{1}{M_j} \right) \frac{1}{d(f_{N_j})}, \quad 0 \leq k \leq q_j,$$

and putting the weights $\Gamma\{\pm t_{j,k}\} := \nu_H[t_{j,k}, t_{j,k+1}]$ for $0 \leq k < q_j$. We let $\Gamma = \nu_H$ outside of the zones of perturbation. The constructed measure Γ is a perturbation of ν_H , because by (29)

$$\max_{0 \leq k < q_j} \left| \frac{t_{j,k+1}}{t_{j,k}} - 1 \right| \leq \frac{M_j(M_j - 1/M_j)}{q_j} \rightarrow 0, \quad j \rightarrow \infty.$$

Therefore, all evaluations from the previous section apply.

4.4 Probabilistic evaluations

Now we fix j for a while and eliminate it from the notation. We will prove that, for the concrete construction in Section 4.3, we have

$$\ln \mathbb{P}\{\max_{1 \leq n \leq N} |S_{1,n}| \leq f_N/3\} \geq -\delta_M f_N^{-1/H} N, \quad (32)$$

$$\mathbb{P}\{\max_{1 \leq n \leq N} |S_{2,n}| \leq f_N/3\} \geq P(M, q), \quad (33)$$

$$\ln \mathbb{P}\{\max_{1 \leq n \leq N} |S_{3,n}| \leq f_N/3\} \geq -\delta_M f_N^{-1/H} N, \quad (34)$$

where δ_M goes to zero when M tends to infinity.

From this, it follows via the correlation inequality (10) that

$$\ln \mathbb{P}\{\max_{1 \leq n \leq N} |S_n| \leq f_N\} \geq -2\delta_M f_N^{-1/H} N + \ln P(M, q).$$

Now we let j vary. By using $M_j \rightarrow \infty$ and (31) we have the required

$$\lim_{j \rightarrow \infty} \frac{\ln \mathbb{P}\{\max_{1 \leq n \leq N_j} |S_n| \leq f_{N_j}\}}{f_{N_j}^{-1/H} N_j} = 0. \quad (35)$$

It remains to justify (32), (33), and (34). We will not repeat in detail the former evaluations leading to the estimates (32) and (34), as the same estimates were shown in (19) and (21). Essentially, it is sufficient to check which properties of the spectral measure and partial sum variances they use.

The bound (32) for the lower frequencies, based on extended Talagrand estimate, uses only (28) in the form

$$\begin{aligned} \mathbb{E} |S_{1,n} - S_{1,m}|^2 &= \int_{|u| \leq \frac{1}{M d(f_N)}} |e^{i(n-m)u} - 1|^2 \tilde{\Gamma}(du) \\ &\leq (n-m)^2 \int_{|u| \leq \frac{1}{M d(f_N)}} u^2 \tilde{\Gamma}(du) \\ &\leq C (n-m)^2 [M d(f_N)]^{2H-2}, \end{aligned}$$

hence,

$$N_c(\varepsilon) \leq C M^{H-1} d(f_N)^{H-1} \frac{N}{\varepsilon},$$

and (12) yields

$$\begin{aligned} \mathbb{P}\{\max_{1 \leq n \leq N} |S_{1,n}| \leq f_N/3\} &\geq \exp\left\{-CM^{H-1}d(f_N)^{H-1} \frac{N}{f_N}\right\} \\ &= \exp\left\{-CM^{H-1} f_N^{-1/H} N\right\}, \end{aligned}$$

as required in (32).

The bound (34) for higher frequencies requires first of all the variance evaluation:

$$\begin{aligned} \mathbb{E} |S_{3,n}|^2 &= \int_{\frac{M}{d(f_N)} \leq |u| \leq \pi} |e^{inu} - 1|^2 \tilde{\Gamma}(du) \\ &\leq 8 \tilde{\Gamma}\left[\frac{M}{d(f_N)}, \pi\right] \\ &\leq C \left(\frac{M}{d(f_N)}\right)^{-2H} = C M^{-2H} f_N^2, \end{aligned}$$

where we used Lemma 10 in the last step.

We also need the increment evaluation

$$\mathbb{E} |S_{3,n_1} - S_{3,n_2}|^2 \leq \mathbb{E} |S_{n_1} - S_{n_2}|^2 = \mathbb{E} |S_{|n_1 - n_2|}|^2 \leq C |n_1 - n_2|^{2H},$$

where we used Proposition 12 in the last step. Then, the estimate (34) follows as before, by an application of the Dudley integral bound.

We pass now to (33) which is the most delicate part. Let

$$\Gamma_k := \Gamma\{t_k\} = \nu_H[t_k, t_{k+1}], \quad 0 \leq k < q.$$

We easily obtain from the definitions of t_k and ν_H that

$$\Gamma_k \leq C \left(\frac{1}{Md(f_N)}\right)^{1-2H} \frac{M}{q d(f_N)} = C M^{2H} d(f_N)^{2H-2} q^{-1}.$$

Furthermore,

$$\begin{aligned} \tilde{\Gamma}_k &:= \tilde{\Gamma}\{t_k\} = \frac{\Gamma_k}{|e^{it_k} - 1|^2} \\ &\leq \frac{C \Gamma_k}{t_k^2} \leq C M^{2+2H} d(f_N)^{2H} q^{-1} = C M^{2+2H} f_N^2 q^{-1}. \end{aligned}$$

The spectral representation yields

$$S_{2,n} = \sum_{k=0}^{q-1} \sqrt{\tilde{\Gamma}_k} (\xi_k \cos(nt_k) + \eta_k \sin(nt_k))$$

where $\{\xi_k, \eta_k\}_{0 \leq k < q}$ is a set of standard Gaussian i.i.d. random variables. It follows that

$$\begin{aligned} \sup_{n \in \mathbb{N}} |S_{2,n}| &\leq \max_k \sqrt{\tilde{\Gamma}_k} \sum_{k=0}^{q-1} (|\xi_k| + |\eta_k|) \\ &\leq C M^{1+H} f_N q^{-1/2} \sum_{k=0}^{q-1} (|\xi_k| + |\eta_k|). \end{aligned}$$

and so

$$\begin{aligned} \mathbb{P} \left\{ \sup_{n \in \mathbb{N}} |S_{2,n}| \leq f_N/3 \right\} &\geq \mathbb{P} \left\{ C M^{1+H} q^{-1/2} \sum_{k=0}^{q-1} (|\xi_k| + |\eta_k|) \leq 1/3 \right\} \\ &:= P(M, q), \end{aligned} \tag{36}$$

as required in (33).

5 Proof of Theorem 4

Upper bound. Define the matrix $K \in \mathbb{R}^{N \times N}$ by $K_{\ell, m} := \mathbb{E} \xi_\ell \xi_m$ for $\ell, m = 1, \dots, N$ and the function $\kappa(x_1, x_2, \dots, x_N) := (x_1, x_1 + x_2, \dots, x_1 + \dots + x_N)$.

If (9) holds, then $\det K > 0$ and we have

$$\mathbb{P} \left\{ \max_{1 \leq n \leq N} |S_n| \leq f_N \right\} = \int_{\{\|\kappa(x)\|_\infty \leq f_N\}} \frac{1}{(2\pi)^{N/2} \sqrt{\det K}} e^{-\langle x, K^{-1}x \rangle / 2} dx.$$

Since K is non-negative definite, it is true that $\langle x, K^{-1}x \rangle \geq 0$ for all $x \in \mathbb{R}^N$, so that we get the following upper bound

$$\mathbb{P} \left\{ \max_{1 \leq n \leq N} |S_n| \leq f_N \right\} \leq \frac{\text{vol}\{\|\kappa(x)\|_\infty \leq f_N\}}{(2\pi)^{N/2} \sqrt{\det K}} = \frac{(2f_N)^N}{(2\pi)^{N/2} \sqrt{\det K}}.$$

By the Szegő limit theorem [2, 8],

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln \det K = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln[2\pi p(u)] du,$$

where p is the density of the absolutely continuous part of the spectral measure. If the integral on the right hand side is finite, we get that

$$\ln \mathbb{P} \left\{ \max_{1 \leq n \leq N} |S_n| \leq f_N \right\} \leq N \ln f_N - N \left[\ln \pi + \frac{1}{4\pi} \int_{-\pi}^{\pi} \ln p(u) du \right] + o(N).$$

Lower bound. First observe that

$$\{x : \|\kappa(x)\|_\infty \leq f_N\} \subseteq \{x : \|x\|_\infty \leq 2f_N\}. \tag{37}$$

Write $(\lambda_j)_{j=1,\dots,N} = (\lambda_j^{(N)})_{j=1,\dots,N}$ for the eigenvalues of $K = K^{(N)}$ and note that a covariance matrix K is diagonalizable (since it is symmetric), say $K = Q^T D Q$ with orthonormal matrix Q and diagonal matrix D . Therefore, for $x \in \mathbb{R}^N$ it is true that

$$\begin{aligned} \langle x, K^{-1}x \rangle &= \langle x, Q^T D^{-1} Q x \rangle = \langle Qx, D^{-1} Qx \rangle = \sum_{j=1}^N \lambda_j^{-1} (Qx)_j^2 \\ &\leq \max_{j=1,\dots,N} \lambda_j^{-1} \cdot \sum_{j=1}^N (Qx)_j^2 \\ &= \frac{1}{\min_{j=1,\dots,N} \lambda_j} \cdot \|Qx\|_2^2 \\ &= \frac{1}{\min_{j=1,\dots,N} \lambda_j} \cdot \|x\|_2^2 \\ &\leq \frac{1}{\min_{j=1,\dots,N} \lambda_j} \cdot N \|x\|_\infty^2. \end{aligned}$$

In particular, for x from the sets in (37),

$$\langle x, K^{-1}x \rangle \leq \frac{1}{\min_{j=1,\dots,N} \lambda_j} \cdot N \cdot (2f_N)^2.$$

In order to estimate the minimum of the eigenvalues, recall from linear algebra that

$$\min_j \lambda_j = \min_{x \in \mathbb{R}^N} \frac{\langle x, Kx \rangle}{\langle x, x \rangle}.$$

Now, note that by the spectral representation

$$\langle x, Kx \rangle = \mathbb{E} \left| \sum_{k=1}^N x_k \xi_k \right|^2 = \int_{[-\pi, \pi)} \left| \sum_{k=1}^N x_k e^{iku} \right|^2 \mu(du)$$

and by the same formula with K replaced by the unit matrix,

$$\langle x, x \rangle = \frac{1}{2\pi} \int_{[-\pi, \pi)} \left| \sum_{k=1}^N x_k e^{iku} \right|^2 du.$$

Fix $\delta > 0$. Let us denote by (\tilde{S}_n) the sequence of partial sums corresponding to the spectral measure $\tilde{\mu} := \mu + \delta\Lambda$ where Λ is the Lebesgue measure. Let \tilde{K} denote the corresponding covariance matrix and $(\tilde{\lambda}_j)_{j=1,\dots,N}$ its eigenvalues.

Using the last three observations for \tilde{K} and $(\tilde{\lambda}_j)$, we get that

$$\min_j \tilde{\lambda}_j = \min_{x \in \mathbb{R}^N} \frac{\int_{[-\pi, \pi)} \left| \sum_{k=1}^N x_k e^{iku} \right|^2 \tilde{\mu}(du)}{\frac{1}{2\pi} \int_{[-\pi, \pi)} \left| \sum_{k=1}^N x_k e^{iku} \right|^2 du} \geq 2\pi\delta.$$

Now we can proceed similarly to the upper bound:

$$\begin{aligned}
\mathbb{P}\{\max_{1 \leq n \leq N} |\tilde{S}_n| \leq f_N\} &= \int_{\{\|\kappa(x)\|_\infty \leq f_N\}} \frac{1}{(2\pi)^{N/2} \sqrt{\det \tilde{K}}} e^{-\langle x, \tilde{K}^{-1}x \rangle/2} dx \\
&\geq \int_{\{\|\kappa(x)\|_\infty \leq f_N\}} \frac{1}{(2\pi)^{N/2} \sqrt{\det \tilde{K}}} e^{-Nf_N^2/(\pi\delta)} dx \\
&= \frac{(2f_N)^N}{(2\pi)^{N/2} \sqrt{\det \tilde{K}}} e^{-Nf_N^2/(\pi\delta)}. \tag{38}
\end{aligned}$$

By using Anderson's inequality, cf. (18), and applying again the Szegő limit theorem for handling $\det \tilde{K}$ we see that

$$\begin{aligned}
\ln \mathbb{P}\{\max_{1 \leq n \leq N} |S_n| \leq f_N\} &\geq \ln \mathbb{P}\{\max_{1 \leq n \leq N} |\tilde{S}_n| \leq f_N\} \\
&\geq N \ln f_N - N \left[\ln \pi + \frac{1}{4\pi} \int_{-\pi}^{\pi} \ln(p(u) + \delta) du + o(1) - \frac{f_N^2}{\pi\delta} \right]. \tag{39}
\end{aligned}$$

By using $f_N \rightarrow 0$ we have $f_N^2/\delta = o(1)$ and the first claim of the theorem,

$$\liminf_{N \rightarrow \infty} \frac{\ln \mathbb{P}\{\max_{1 \leq n \leq N} |S_n| \leq f_N\}}{N \ln f_N^{-1}} \geq -1,$$

follows from (39).

Inequality (39) also yields

$$\liminf_{N \rightarrow \infty} \frac{\ln \mathbb{P}\{\max_{1 \leq n \leq N} |S_n| \leq f_N\} - N \ln f_N}{N} \geq -\ln \pi - \frac{1}{4\pi} \int_{-\pi}^{\pi} \ln(p(u) + \delta) du.$$

In the case $\int_{-\pi}^{\pi} \ln p(u) du > -\infty$, letting here $\delta \rightarrow 0$ shows the second claim of the theorem.

6 Constant boundary

Let $M_N = \max_{1 \leq n \leq N} |S_n|$. We want to prove in particular that for any constant $f > 0$ there exists the limit

$$\lim_{N \rightarrow \infty} \frac{\ln \mathbb{P}\{M_N \leq f\}}{N} \in (-\infty, 0]. \tag{40}$$

First we prove an intermediate result (Proposition 13) showing that the limit exists for a certain class of strictly increasing functions. Then we show that the limit exists for constants (Proposition 15) before finally proving the full main result (Theorem 6).

Proposition 13 *Let (f_N) be a strictly increasing, positive sequence satisfying the growth condition*

$$\sum_{m=1}^{\infty} r_m < \infty \quad (41)$$

where

$$r_m := 2^{-m} \max_{2^m \leq b \leq 2^{m+1}} |\ln(f_b - f_{7b/8})|.$$

Then there exists the limit

$$\lim_{N \rightarrow \infty} \frac{\ln \mathbb{P}\{M_N \leq f_N\}}{N} \in (-\infty, 0]. \quad (42)$$

Remark 14 Condition (41) holds, for example, for the boundaries of the type $f_N = f - cN^{-q}$, $N > N_0$ for any parameters $f, c, q > 0$.

Proof: Let us set

$$L_N := \frac{\ln \mathbb{P}\{M_N \leq f_N\}}{N}, \quad N \in \mathbb{N}.$$

Let $\Delta_m := \{N \in \mathbb{N} : 2^m \leq N \leq 2^{m+1}\}$ be the binary blocks and denote

$$\mathcal{I}_m := \min_{N \in \Delta_m} L_N, \quad \mathcal{M}_m := \max_{N \in \Delta_m} L_N.$$

It is sufficient for us to prove that there exist equal limits

$$\lim_{m \rightarrow \infty} \mathcal{M}_m = \lim_{m \rightarrow \infty} \mathcal{I}_m \in (-\infty, 0]. \quad (43)$$

We first prove that the second limit exists.

For any $N_1, N_2 \in \mathbb{N}$, $f, \delta > 0$, the correlation inequality yields

$$\mathbb{P}\{M_{N_1+N_2} \leq f + \delta\} \geq \mathbb{P}\{M_{N_1} \leq f + \delta\} \mathbb{P}\{|S_{N_1}| \leq \delta\} \mathbb{P}\{M_{N_2} \leq f\}. \quad (44)$$

This will be our main tool along the proof.

For any $m \in \mathbb{N}$ let us take $b \in \Delta_{m+1}$ such that $L_b = \mathcal{I}_{m+1}$ and represent it in the form $b = b_1 + b_2$ with $b_1 = b_2 = b/2$ for even b and $b_1 = (b+1)/2, b_2 = (b-1)/2$ for odd b . In any case we have $b_1, b_2 \in \Delta_m$. By applying (44) with $N_1 = b_1, N_2 = b_2$ and $f = f_{b_2}, \delta = f_b - f_{b_2}$ we obtain

$$\begin{aligned} \mathbb{P}\{M_b \leq f_b\} &\geq \mathbb{P}\{M_{b_1} \leq f_b\} \mathbb{P}\{|S_{b_1}| \leq f_b - f_{b_2}\} \mathbb{P}\{M_{b_2} \leq f_{b_2}\} \\ &\geq \mathbb{P}\{M_{b_1} \leq f_{b_1}\} \mathbb{P}\{cb_1|\mathcal{N}| \leq f_b - f_{7b/8}\} \mathbb{P}\{M_{b_2} \leq f_{b_2}\} \\ &\geq \mathbb{P}\{M_{b_1} \leq f_{b_1}\} c \min\left\{\frac{f_b - f_{7b/8}}{b_1}, 1\right\} \mathbb{P}\{M_{b_2} \leq f_{b_2}\}, \end{aligned}$$

where \mathcal{N} is a standard normal random variable. By taking logarithms, this leads to

$$\begin{aligned} bL_b &\geq b_1L_{b_1} + b_2L_{b_2} - c - \left| \ln \frac{f_b - f_{7b/8}}{b_1} \right| \\ &\geq b_1L_{b_1} + b_2L_{b_2} - c - |\ln(f_b - f_{7b/8})| - \ln b_1 \end{aligned}$$

and we obtain

$$\begin{aligned} \mathcal{I}_{m+1} &= L_b \geq \frac{b_1}{b}L_{b_1} + \frac{b_2}{b}L_{b_2} - \frac{1}{b} [c + |\ln(f_b - f_{7b/8})| + \ln b_1] \\ &\geq \mathcal{I}_m - cr'_m \end{aligned} \tag{45}$$

where $r'_m = r_m + m2^{-m}$. It follows from (45) that for any $m_0 \in \mathbb{N}$

$$\liminf_{m \rightarrow \infty} \mathcal{I}_m \geq \mathcal{I}_{m_0} - c \sum_{m=m_0}^{\infty} r'_m$$

where the series is convergent by (41). We observe from this inequality that $\liminf_{m \rightarrow \infty} \mathcal{I}_m$ is not equal to $-\infty$. Furthermore, taking limsup in the right hand side we obtain

$$\liminf_{m \rightarrow \infty} \mathcal{I}_m \geq \limsup_{m_0 \rightarrow \infty} \mathcal{I}_{m_0}.$$

Therefore, the existence of $\lim_{m \rightarrow \infty} \mathcal{I}_m$ and its finiteness are now proved. It remains to prove that

$$\limsup_{m \rightarrow \infty} \mathcal{M}_m \leq \lim_{m \rightarrow \infty} \mathcal{I}_m. \tag{46}$$

For any $m \in \mathbb{N}$ choose $a \in \Delta_{m-1}$, $b \in \Delta_{m+1}$ such that $L_a = \mathcal{M}_{m-1}$ and $L_b = \mathcal{I}_{m+1}$. Notice that

$$2^{m+2} \geq b \geq b - a \geq 2^{m+1} - 2^m = 2^m,$$

hence, $b - a \in \Delta_m \cup \Delta_{m+1}$. We also have

$$\frac{b-a}{b} = 1 - \frac{a}{b} \leq \frac{7}{8}.$$

By applying (44) with $N_1 = a$, $N_2 = b - a$, $f = f_{b-a}$, $\delta = f_b - f_{b-a}$ we obtain

$$\begin{aligned} \mathbb{P}\{M_b \leq f_b\} &\geq \mathbb{P}\{M_a \leq f_b\} \mathbb{P}\{|S_a| \leq f_b - f_{b-a}\} \mathbb{P}\{M_{b-a} \leq f_{b-a}\} \\ &\geq \mathbb{P}\{M_a \leq f_a\} \mathbb{P}\{|S_a| \leq f_b - f_{b-a}\} \mathbb{P}\{M_{b-a} \leq f_{b-a}\}. \end{aligned}$$

By taking logarithms we get

$$\begin{aligned}
b\mathcal{I}_{m+1} &= bL_b \geq aL_a + (b-a)L_{b-a} + \ln \mathbb{P}(ca|\mathcal{N}| \leq f_b - f_{b-a}) \\
&\geq a\mathcal{M}_{m-1} + (b-a) \min\{\mathcal{I}_m, \mathcal{I}_{m+1}\} + \ln \mathbb{P}(ca|\mathcal{N}| \leq f_b - f_{7b/8}) \\
&\geq a\mathcal{M}_{m-1} + (b-a) \min\{\mathcal{I}_m, \mathcal{I}_{m+1}\} - c - \left| \ln \left[\frac{f_b - f_{7b/8}}{a} \right] \right|.
\end{aligned}$$

We may rewrite this inequality as

$$\begin{aligned}
a\mathcal{M}_{m-1} &\leq b\mathcal{I}_{m+1} - (b-a) \min\{\mathcal{I}_m, \mathcal{I}_{m+1}\} + c + \left| \ln \left[\frac{f_b - f_{7b/8}}{a} \right] \right| \\
&= a\mathcal{I}_{m+1} + (b-a)[\mathcal{I}_{m+1} - \min\{\mathcal{I}_m, \mathcal{I}_{m+1}\}] + c + \left| \ln \left[\frac{f_b - f_{7b/8}}{a} \right] \right| \\
&\leq a\mathcal{I}_{m+1} + (b-a)|\mathcal{I}_{m+1} - \mathcal{I}_m| + c + \left| \ln \left[\frac{f_b - f_{7b/8}}{a} \right] \right|.
\end{aligned}$$

Dividing by a and using $(b-a)/a \leq 8$ yields

$$\begin{aligned}
\mathcal{M}_{m-1} &\leq \mathcal{I}_{m+1} + \frac{b-a}{a} |\mathcal{I}_{m+1} - \mathcal{I}_m| + \frac{1}{a} (c + |\ln(f_b - f_{7b/8})| + \ln a) \\
&\leq \mathcal{I}_{m+1} + 8 |\mathcal{I}_{m+1} - \mathcal{I}_m| + cr'_{m+1}.
\end{aligned}$$

Since $\lim_{m \rightarrow \infty} (\mathcal{I}_{m+1} - \mathcal{I}_m) = 0$ and $\lim_{m \rightarrow \infty} r'_m = 0$, taking the limit yields (46). \square

Now we may prove the main result of this section.

Proposition 15 *For every constant $f > 0$ the following limit exists:*

$$\lim_{N \rightarrow \infty} \frac{\ln \mathbb{P}\{M_N \leq f\}}{N} \in (-\infty, 0]. \quad (47)$$

Proof: We shall make use of the log-concavity of Gaussian measures. For any sequence (f_N) let

$$\mathcal{L}((f_N)) := \lim_{N \rightarrow \infty} \frac{\ln \mathbb{P}\{M_N \leq f_N\}}{N}$$

if this limit exists. We fix $f > 0$ and a sequence $\delta_N = \frac{1}{N}$. Notice that for any $C > 0$ the sequence $f_N^{(C)} := f - C\delta_N$ satisfies assumption of Proposition 13 (see Remark 14). Therefore, $\mathcal{L}((f_N^{(C)}))$ is well defined. Finally, let

$$\mathcal{I}_f := \liminf_{N \rightarrow \infty} \frac{\ln \mathbb{P}\{M_N \leq f\}}{N}; \quad \mathcal{M}_f := \limsup_{N \rightarrow \infty} \frac{\ln \mathbb{P}\{M_N \leq f\}}{N}.$$

We shall prove that

$$\mathcal{I}_f = \mathcal{M}_f = \mathcal{L}((f_N^{(1)})).$$

Since we obviously have

$$\mathcal{L}((f_N^{(1)})) \leq \mathcal{I}_f \leq \mathcal{M}_f,$$

it remains to prove that $\mathcal{M}_f \leq \mathcal{L}((f_N^{(1)}))$.

By the well known log-concavity of Gaussian measures ([3]), for each $N \in \mathbb{N}$ the function $r \mapsto \ln \mathbb{P}(M_N \leq r)$ is concave. In particular, for every $C > 1$ we have

$$\begin{aligned} & \ln \mathbb{P}\{M_N \leq f\} - \ln \mathbb{P}\{M_N \leq f - \delta_N\} \\ & \leq \frac{1}{C} (\ln \mathbb{P}\{M_N \leq f\} - \ln \mathbb{P}\{M_N \leq f - C\delta_N\}). \end{aligned}$$

Dividing by N and taking \limsup over N yields

$$\mathcal{M}_f - \mathcal{L}((f_N^{(1)})) \leq \frac{1}{C} \left(\mathcal{M}_f - \mathcal{L}((f_N^{(C)})) \right) \leq \frac{1}{C} (\mathcal{M}_f - \mathcal{L}((f/2 - \delta_N))).$$

Recall that $0 \geq \mathcal{M}_f \geq \mathcal{L}((f/2 - \delta_N)) > -\infty$ by Proposition 13. Therefore, letting $C \rightarrow \infty$ yields $\mathcal{M}_f - \mathcal{L}((f_N^{(1)})) \leq 0$, as required. \square

Proof of Theorem 6: Let the function $\mathfrak{C} : (0, \infty) \mapsto (-\infty, 0]$ be defined, according to (47), as

$$\mathfrak{C}(f) := \lim_{N \rightarrow \infty} \frac{\ln \mathbb{P}\{M_N \leq f\}}{N}.$$

As a pointwise limit of concave functions, $\mathfrak{C}(\cdot)$ is itself concave, hence it is continuous.

Now take a sequence (f_N) satisfying our Theorem's assumption and denote $f := \lim_{N \rightarrow \infty} f_N \in (0, \infty)$. It is obvious that for any $\delta \in (0, f)$

$$\mathfrak{C}(f - \delta) \leq \liminf_{N \rightarrow \infty} \frac{\ln \mathbb{P}\{M_N \leq f_N\}}{N} \leq \limsup_{N \rightarrow \infty} \frac{\ln \mathbb{P}\{M_N \leq f_N\}}{N} \leq \mathfrak{C}(f + \delta).$$

By letting $\delta \rightarrow 0$ and using the continuity of $\mathfrak{C}(\cdot)$ we obtain

$$\mathfrak{C}(f) \leq \liminf_{N \rightarrow \infty} \frac{\ln \mathbb{P}\{M_N \leq f_N\}}{N} \leq \limsup_{N \rightarrow \infty} \frac{\ln \mathbb{P}\{M_N \leq f_N\}}{N} \leq \mathfrak{C}(f).$$

It follows that

$$\lim_{N \rightarrow \infty} \frac{\ln \mathbb{P}\{M_N \leq f_N\}}{N} = \mathfrak{C}(f).$$

It remains to confirm that this limit is strictly negative assuming that Kolmogorov criterion holds. In this case $\sigma^2 := \text{Var}(\xi_1 | \xi_0, \xi_{-1}, \xi_{-2}, \dots) > 0$

(see [5]). We obtain with Anderson's inequality (cf. (18)):

$$\begin{aligned}
& \mathbb{P}\{\max_{1 \leq n \leq N} |S_n| \leq f\} \\
&= \mathbb{E}[\mathbb{P}\{\max_{1 \leq n \leq N} |S_n| \leq f | \xi_{N-1}, \xi_{N-2}, \dots\}] \\
&= \mathbb{E}[\mathbb{1}_{\max_{1 \leq n \leq N-1} |S_n| \leq f} \mathbb{P}\{|S_{N-1} + \xi_N| \leq f | \xi_{N-1}, \xi_{N-2}, \dots\}] \\
&\leq \mathbb{E}[\mathbb{1}_{\max_{1 \leq n \leq N-1} |S_n| \leq f} \mathbb{P}\{|\sigma \mathcal{N}| \leq f\}] \\
&= \mathbb{P}\{\max_{1 \leq n \leq N-1} |S_n| \leq f\} \mathbb{P}\{|\sigma \mathcal{N}| \leq f\} \\
&\leq \dots \\
&\leq \mathbb{P}\{|\sigma \mathcal{N}| \leq f\}^N,
\end{aligned}$$

where \mathcal{N} is a standard normal random variable. This shows that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln \mathbb{P}\{\max_{1 \leq n \leq N} |S_n| \leq f\} \leq \ln \mathbb{P}\{|\sigma \mathcal{N}| \leq f\} < 0.$$

□

Remark 16 There are various interesting open questions related to the constant $\mathfrak{C}(f)$ from Theorem 6. If we consider it as a function $\mathfrak{C} : (0, \infty) \rightarrow (-\infty, 0]$, we may ask: 1) Is it true that $\lim_{f \rightarrow 0} \mathfrak{C}(f) = -\infty$? 2) Is it true that $\lim_{f \rightarrow \infty} \mathfrak{C}(f) = 0$? 3) Is it true that for every singular process $\mathfrak{C}(f) = 0$ for all $f > 0$?

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